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Estimate for the lower bound of rank and the upper bound of eigenvalues norms' sum of given quaternion matrix[☆]

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ABSTRACT

This work is concerned with estimating the lower bound of rank for a given quaternion square matrix, rectangular matrix or normal matrix, and estimating the upper bound for the sum of the eigenvalue norms of a given quaternion matrix. A sufficient condition is provided to confirm whether a given quaternion matrix is nonsingular. It sharpens some results due to Tu Bo-xun and Semyon Aranovich Geršchgorin and Schur, respectively. Some examples are provided to show the effectiveness of our results.

1. Introduction

In recent years, the problems over quaternion division algebra have drawn the wide attention of researchers in mathematics, physics and computer science. Many problems of quaternion algebra have been researched, such as the polynomial, the determinant, the system of quaternion matrix equations, etc. It is not only the property of non-commutative multiplication of the quaternion, but also its wide-ranging connection with many applied sciences, such as Quantum Physics, Geostatics, computer graphics, pattern recognition, and space telemetry and so forth. These technologies also need further study of the quaternion algebra.

In algebra, the eigenvalue and rank are always two important aspects of matrix analysis. We found that it has been well estimated for the lower bound of the rank and the upper bound of eigenvalues of a given complex matrix in [1–4]. However, on quaternion division algebra, these problems still need further study. So, in this paper, we try to discuss these problems.

This paper is organized as follows: In Section 2, we introduce some estimation methods for the rank and standard eigenvalues (the standard eigenvalue is also called a right eigenvalue) of a given quaternion matrix. In Section 3, based on the generalized Geršchgorin Theorem, we give three methods to estimate the left eigenvalues of a given quaternion matrix.

Throughout this paper, we adopt following notations and terminologies: R and C are the sets of real and complex numbers, respectively, H denotes the set of quaternions. $H^{n\times n}$ denotes the set of $n \times n$ quaternion matrices.

For any quaternion a, let $N(a) = \sqrt{a\bar{a}} = \sqrt{\bar{a}a} = (a_0^2 + a_1^2 + a_2^2 + a_3^2)^{1/2}$ be called the norm of a. For any a quaternion matrix A, $r(A)$ denotes its rank and $\|A\|$ denotes its determinant, $\mathrm{tr}(A)$ denotes its trace, A^* denotes its conjugate-transpose. $U^{n\times n}$ denotes the set of all quaternion unitary matrices.

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2. Estimate for the lower bound of rank and the upper bound of right eigenvalues norms' sum of quaternion matrix

In this section, we first establish some theorems to estimate the lower bound of rank and the upper bound for the sum of right eigenvalues norms of a given quaternion matrix by use of the partition matrix method.

Lemma 2.1. *Let $A, B \in H^{n\times n}$ where A is a unitary similar matrix of B [5], then $\|A\|^2 = \|B\|^2$.*

Proof. Since A is unitary similar to B, there exists a quaternion matrix $U \in U^{n\times n}$ such that $B = U^*AU$, thus $BB^* = U^*AUU^*A^*U = U^*AA^*U$. It follows that $\operatorname{tr} AA^* = \operatorname{tr} BB^*$.

Moreover, $\operatorname{tr} AA^* = \|A\|^2$ and $\operatorname{tr} BB^* = \|B\|^2$, so $\|A\|^2 = \|B\|^2$.

So, the lemma is proved. □

Theorem 2.1. *Let $M \in H^{n\times n}$. If M can be partitioned into $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ (where principal submatrix A is $k \times k$ $(1 \le k \le n-1)$, submatrix D is $(n-k)\times(n-k)$ and $B \ne 0$, $C \ne 0$), and then let $l_k = \operatorname{tr}(AA^*) + \operatorname{tr}(DD^*) + 2\sqrt{\operatorname{tr}(BB^*)\operatorname{tr}(CC^*)}$ and $l = \min_k l_k$, then inequality $r(M) \ge \frac{1}{l}(\operatorname{Re}(\operatorname{tr} M))^2$ holds.*

Proof. Since $M \in H^{n\times n}$, then there exists a unitary matrix U such that

$$U^*MU = \begin{bmatrix} \lambda_1 & q_{12} & \cdots & q_{1n} \\ 0 & \lambda_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \tag{2.1}$$

where $\lambda_s = \lambda_s^{(1)} + \lambda_s^{(2)} i \in C$, $\lambda_s^{(2)} \ge 0$ $(s = 1, 2, \ldots, n)$ are the standard eigenvalues (right eigenvalues) of M.

We suppose that M has s nonzero standard eigenvalues, then $s = r(U^*MP) = r(M)$ and

$$[\operatorname{Re}(\operatorname{tr} M)]^2 = [\operatorname{Re}(\operatorname{tr} U^*MU)]^2 = \left(\operatorname{Re}\left(\sum_{i=1}^{n} \lambda_i\right)\right)^2 = \left(\sum_{i=1}^{s} \operatorname{Re}\lambda_i\right)^2 \le \left(\sum_{i=1}^{s} |\lambda_i|\right)^2, \tag{2.2}$$

therefore,

$$[\operatorname{Re}(\operatorname{tr} M)]^2 \le s\sum_{i=1}^{s} |\lambda_i|^2 = r(M)\sum_{i=1}^{n} |\lambda_i|^2. \tag{2.3}$$

If we take $\mu_1^2 = \operatorname{tr}(BB^*) > 0$, $\mu_2^2 = \operatorname{tr}(CC^*) > 0$ and construct the following matrix

$$K = \begin{pmatrix} A & \left(\dfrac{\mu_2}{\mu_1}\right)^{1/2} B \\ \left(\dfrac{\mu_1}{\mu_2}\right)^{1/2} C & D \end{pmatrix},$$

then

$$KK^* = \begin{pmatrix} AA^* + \dfrac{\mu_2}{\mu_1}BB^* & * \\ * & \dfrac{\mu_1}{\mu_2}CC^* + DD^* \end{pmatrix}.$$

So

$$\operatorname{tr}(KK^*) = \operatorname{tr}(AA^*) + \operatorname{tr}(DD^*) + 2\mu_1\mu_2 = l_k. \tag{2.4}$$

For $K = \begin{pmatrix} \left(\frac{\mu_2}{\mu_1}\right)^{1/2} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \left(\frac{\mu_1}{\mu_2}\right)^{1/2} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}$ (where I_k $(1 \le k \le n-1)$ are $k \times k$ identity matrices), we can easily find that M is unitary similar to K. Therefore $\lambda_1, \lambda_2, \ldots, \lambda_n$ are also the right eigenvalues of K. Then there exists a unitary matrix U_1 such that

$$U_1^*KU_1 = \begin{bmatrix} \lambda_1 & a_{12} & \cdots & a_{1n} \\ 0 & \lambda_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

We have

$$U_1^*K^*U_1 = \begin{bmatrix} \overline{\lambda_1} & 0 & \cdots & 0 \\ \overline{a_{12}} & \overline{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{\lambda_n} \end{bmatrix},$$

$$U_1^*KK^*U_1 = (U_1^*KU_1)(U_1^*K^*U_1) = \begin{bmatrix} |\lambda_1|^2 + \sum_{j=2}^{n} |a_{1j}|^2 & * & \cdots & * \\ * & |\lambda_2|^2 + \sum_{j=3}^{n} |a_{2j}|^2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & |\lambda_n|^2 \end{bmatrix}. \tag{2.5}$$

It follows that

$$\operatorname{tr}(KK^*) = \operatorname{tr}(U^*KK^*U) \geq \sum_{i=1}^{n} |\lambda_i|^2. \tag{2.6}$$

From (2.3) and (2.6), we have (2.7).

$$[\operatorname{Re}(\operatorname{tr} M)]^2 \leq r(M)\operatorname{tr}(KK^*) = l_k r(M) \quad (1 \leq k \leq n-1). \tag{2.7}$$

So,

$$(\operatorname{Re}(\operatorname{tr} M))^2 \leq l r(M). \tag{2.8}$$

Thus, the theorem is proved. □

Corollary 2.1. *Let $M \in H^{n\times n}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be standard eigenvalues (right eigenvalues) of M, then inequality $\sum_{i=1}^{n} |\lambda_i|^2 \leq \operatorname{tr}(MM^*)$ holds.*

Proof. According to (2.6), it is obvious that $\sum_{i=1}^{n} |\lambda_i|^2 \leq \operatorname{tr}(KK^*) \leq \operatorname{tr}(MM^*)$. □

Theorem 2.1 can be used to confirm whether or not one given quaternion matrix is nonsingular. We have

Corollary 2.2. *Let M and l be defined as the Theorem 2.1, if $(\operatorname{Re}(\operatorname{tr} M))^2 > (n-1)l$, then we can be sure that M is a nonsingular quaternion matrix.*

Proof. From the Theorem 2.1, we have $(\operatorname{Re}(\operatorname{tr} M))^2 \leq l r(M)$, so $r(M) > n-1$. Therefore M is a nonsingular quaternion matrix. □

Example. Let $A = \begin{bmatrix} 3 & 1 & i & 0 \\ 0 & 3 & j & -1 \\ k & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 \end{bmatrix}$, we know that $l = \min_k l_k = l_2 = 21+4\sqrt{7}$ and $(\operatorname{Re}(\operatorname{tr} A))^2 = 11^2 > 3(21+4\sqrt{7}) = 3l$. By using the Corollary 2.2, we can conclude that A is a nonsingular quaternion matrix. □

The condition of Corollary 2.2 is a sufficient condition for a given quaternion matrix to be nonsingular.

Theorem 2.1 shows the lower bound of the rank of a given square quaternion matrix over quaternion division algebra. Next we will turn our attention to discussing the case for the quaternion matrix to be rectangular.

Theorem 2.2. *Let $M \in H^{m\times n}$ $(m \neq n)$, then $r(M) \geq \frac{(\operatorname{tr}(MM^*))^2}{\operatorname{tr}(MM^*)^2}$.*

Proof. For any a given quaternion matrix $A \in H^{n\times n}$, by using the inequality (2.7), we have

$$r(A) \geq \frac{(\operatorname{Re}\operatorname{tr}(A))^2}{\operatorname{tr}(AA^*)}. \tag{2.9}$$

In (2.9), let MM^* replace A, then it follows that

$$r(MM^*) \geq \frac{(\operatorname{Re}\operatorname{tr}(MM^*))^2}{\operatorname{tr}((MM^*)(MM^*)^*)} = \frac{(\operatorname{tr}(MM^*))^2}{\operatorname{tr}(MM^*)^2}. \tag{2.10}$$

Because $r(M) = r(MM^*)$ [5], so,

$$r(M) \geq \frac{(\mathrm{tr}(MM^*))^2}{\mathrm{tr}(MM^*)^2}. \tag{2.11}$$

Thus, the proof is complete. □

In the following, we will discuss the lower bound of the rank of the quaternion matrix in the normal case.

Theorem 2.3. *Let $M \in H^{n\times n}$ be a given quaternion normal matrix, then $r(M) \geq \frac{(\mathrm{Re}(\mathrm{tr}\, M))^2}{\mathrm{tr}(MM^*)}$.*

Proof. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ (where A is the $k \times k$ principal submatrix of M), then $MM^* = \begin{pmatrix} AA^* + BB^* & * \\ * & * \end{pmatrix}$ and $M^*M = \begin{pmatrix} A^*A + C^*C & * \\ * & * \end{pmatrix}$. Since $M \in H^{n\times n}$ is a quaternion normal matrix, that is, $MM^* = M^*M$, we have

$$AA^* + BB^* = A^*A + C^*C. \tag{2.12}$$

And then,

$$\mathrm{tr}(BB^*) = \mathrm{tr}(C^*C) = \mathrm{tr}(CC^*). \tag{2.13}$$

Moreover,

$$\begin{aligned} l_k &= \mathrm{tr}(AA^*) + \mathrm{tr}(DD^*) + 2\sqrt{\mathrm{tr}(BB^*)\mathrm{tr}(CC^*)} \\ &= \mathrm{tr}(AA^*) + \mathrm{tr}(DD^*) + 2\mathrm{tr}(BB^*) \\ &= \mathrm{tr}(AA^*) + \mathrm{tr}(DD^*) + \mathrm{tr}(BB^*) + \mathrm{tr}(CC^*) \\ &= \mathrm{tr}(MM^*). \end{aligned} \tag{2.14}$$

Notice that the l_k cannot be changed by that of k, that is $l = \mathrm{tr}(MM^*)$ always holds. So, according to Theorem 2.1, Theorem 2.3 is proved. □

We now turn our attention to estimating the upper bounds of the sums of the absolute values of the real parts and the imaginary parts of the right eigenvalues of the given quaternion matrix. We have the following fact.

Theorem 2.4. *Let $\lambda_s = \lambda_s^{(1)} + \lambda_s^{(2)} i \in C$ $(s = 1, \ldots, n)$ be the standard eigenvalues of a given quaternion matrix A, then $\sum_{s=1}^n |\lambda_s^{(1)}| \leq \left\| \frac{A+A^*}{2} \right\|^2$ and $\sum_{s=1}^n |\lambda_s^{(2)}| \leq \left\| \frac{A-A^*}{2i} \right\|^2$.*

Proof. According to the generalized Schur Theorem [1], there exists $U \in U^{n\times n}$ such that

$$U^*AU = \begin{bmatrix} \lambda_1 & q_{12} & \cdots & q_{1n} \\ 0 & \lambda_2 & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

(where $\lambda_s = \lambda_s^{(1)} + \lambda_s^{(2)} i \in C$ $(s = 1, \ldots, n)$ are the standard eigenvalues of A). So, we have

$$\|A\|^2 = \left\|U^*AU\right\|^2 = \sum_{s=1}^n |\lambda_s|^2 + \sum_{2\leq i<j\leq n} \left|q_{ij}\right|^2. \tag{2.15}$$

Moreover,

$$U^*\left(\frac{A+A^*}{2}\right)U = \begin{bmatrix} \lambda_1^{(1)} & \frac{1}{2}q_{12} & \cdots & \frac{1}{2}q_{1n} \\ \frac{1}{2}\overline{q_{12}} & \lambda_2^{(1)} & \cdots & \frac{1}{2}q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\overline{q_{1n}} & \frac{1}{2}\overline{q_{2n}} & \cdots & \lambda_n^{(1)} \end{bmatrix}. \tag{2.16}$$

By the Lemma 2.1, we obtain

$$\left\|\frac{A+A^*}{2}\right\|^2 = \sum_{s=1}^n \left|\lambda_s^{(1)}\right|^2 + \frac{1}{2}\sum_{2\leq i<j\leq n} \left|q_{ij}\right|^2. \tag{2.17}$$

Similarly, we have

$$\left\| \frac{A-A^*}{2i} \right\|^2 = \sum_{s=1}^{n} \left|\lambda_s^{(2)}\right|^2 + \frac{1}{2} \sum_{2\le i<j\le n} \left|q_{ij}\right|^2 . \tag{2.18}$$

Hence

$$\sum_{s=1}^{n} \left|\lambda_s^{(1)}\right|^2 = \left\| \frac{A+A^*}{2} \right\|^2 - \frac{1}{2} \sum_{2\le i<j\le n} \left|q_{ij}\right|^2 = \left\| \frac{A+A^*}{2} \right\|^2 + \frac{1}{2} \left(\sum_{s=1}^{n} |\lambda_s|^2 - \|A\|^2 \right), \tag{2.19}$$

$$\sum_{s=1}^{n} \left|\lambda_s^{(2)}\right|^2 = \left\| \frac{A-A^*}{2i} \right\|^2 - \frac{1}{2} \sum_{2\le i<j\le n} \left|q_{ij}\right|^2 = \left\| \frac{A-A^*}{2i} \right\|^2 + \frac{1}{2} \left(\sum_{s=1}^{n} |\lambda_s|^2 - \|A\|^2 \right). \tag{2.20}$$

By using the Corollary 2.1, it follows that

$$\sum_{i=1}^{n} |\lambda_s^{(1)}| \le \left\| \frac{A+A^*}{2} \right\|^2 \quad \text{and} \quad \sum_{i=1}^{n} |\lambda_s^{(2)}| \le \left\| \frac{A-A^*}{2i} \right\|^2 .$$

So, the proof is complete. □

3. Estimate for the upper bound of the sum of the left eigenvalue norms for a given quaternion matrix

The discussion above shows the upper bound estimation of the right eigenvalues' real parts and imaginary parts for a given quaternion matrix. In the following, we will give some estimations of the upper bounds of the other type of eigenvalue.

Notice that the Geršchgorin theorem has been generalized from the complex field to quaternion division algebra [6,7]. The primary role of the Geršchgorin theorem on a complex field and the generalized Geršchgorin theorem over quaternion algebra is applied to locate eigenvalues. However, we find that it also has many other important roles for the estimation of eigenvalues. So, in the next section, we will use the generalized Geršchgorin theorem to discuss some estimation methods for left eigenvalues of quaternion matrices.

First of all, it is worthy of note that we only consider the estimation problems of left eigenvalues under the condition that no left eigenvalues of a given quaternion matrix are repeated. If in a quaternion matrix there are repeated eigenvalues, either we do not know which eigenvalue is appropriate to be repeated, or we know the fact in advance, in which case we know each eigenvalue exactly and would not be interested in estimating it. In addition, our results will show that, if in a given quaternion matrix there are repeated eigenvalues, then this case can be discussed on a similar plan also.

Definition 3.1. Let $A \in H^{n\times n}$. A quaternion λ is called a left eigenvalue of A if there exists a n-dimensional nonzero quaternion column vector $X = \{x_1, x_2, \dots, x_n\}^T$ such that $AX = \lambda X$.

Definition 3.2. Let a be a given quaternion and ε be a positive number. Then the set $\Omega = \{z | N(z-a) \le \varepsilon\}$ is called a generalized spherical neighbourhood with centre a and radius ε.

Article [6] has proved the generalized Geršchgorin theorem over quaternion algebra, to use it here, we give it as a lemma.

Lemma 3.1 (*[6]*). *Let $A = (a_{ij}) \in H^{n\times n}$. Any left eigenvalue λ of A has to lie within at least one of n spherical neighbourhoods $\Omega_i = \{z | N(z-a_{ii}) \le P_i, (i = 1, 2, \dots, n)\}$. That is $\lambda \in \bigcup_{i=1}^{n} \Omega_i(A) = \bigcup_{i=1}^{n} \{z | N(z-a_{ii})\} \le P_i$. Where $P_i = \sum_{j=1, j\ne i}^{n} N(a_{ij})$.*

Through this proof of the generalized Geršchgorin theorem, we can turn it into following form. It will be used in this paper.

Lemma 3.2. *Let $A = (a_{ij}) \in H_{n\times n}$. Then the left eigenvalues of A have to lie within at least one of n generalized spherical neighbourhoods $\Omega_i(A) = \{z : N(z-a_{ii}) \le \sqrt{n-1}R_i\}$ $(i = 1, 2, \dots, n)$, where $R_i = \sqrt{\sum_{j\ne i}^{n} N^2(a_{ij})}$.*

The proof of Lemma 3.2 can be found in the literature [6] and hence it is omitted here.

It can be seen that all the left eigenvalues of A lie within the union of n generalized spherical neighbourhoods with centres a_{ii} and radii $\sqrt{n-1}R_i$ $(i = 1, 2, \dots, n)$, respectively.

We next state and prove our main results.

Theorem 3.1. *Let $A = (a_{ij}) \in H^{n\times n}$. If λ_i $(i = 1, 2, \dots, n)$ are left eigenvalues of A which lie within n distinct generalized spherical neighbourhoods, respectively, then $\sum_{i=1}^{n} N(\lambda_i) \le \sum_{i=1}^{n} \sum_{j=1}^{n} N(a_{ij})$.*

Proof. Since λ_i $(i = 1, \ldots, n)$ are left eigenvalues of A which lie within n distinct generalized spherical neighbourhoods, respectively, without loss of generality, we assume that $\lambda_i \in \Omega_i = \{z \mid N(z - a_{ii}) \le R_i\}$, where $R_i = \sum_{\substack{j=1\\ j\ne i}}^{n} N(a_{ij})$, $\Omega_i(A) \ne \Omega_j(A)$, $i, j = 1, \ldots, n$, $i \ne j$.

By virtue of Lemma 3.1, we have $N(\lambda_i - a_{ii}) \le \sum_{\substack{j=1\\ j\ne i}}^{n} N(a_{ij})$ $(i = 1, 2, \ldots, n)$. Hence, we can derive that, $N(\lambda_i) \le N(a_{ii}) + \sum_{\substack{j=1\\ j\ne i}}^{n} N(a_{ij})$, $(i = 1, 2, \ldots, n)$. Therefore, $\sum_{i=1}^{n} N(\lambda_i) \le \sum_{i=1}^{n}\sum_{j=1}^{n} N(a_{ij})$.

Thus, the proof is complete. □

Note that the Theorem 3.1 is similar to Schur's well-known inequality [8] which is used to describe the relation of the absolute values of the eigenvalues and the absolute values of entries of a given complex matrix. However, whether it is the form and content or the method of proof, Theorem 3.1 doesn't repeat Schur's inequality. (Schur's inequality is that $\sum_{i=1}^{n} |\lambda_i|^2 \le \sum_{\substack{i=1\\ j=1}}^{n} |a_{ij}|^2$, where, $(a_{ij}) \in C_{n\times n}$ denotes a given complex matrix).

Next, we give the second form of the upper bounds of all norms' sum of n left eigenvalues of a given quaternion matrix A.

Theorem 3.2. *Let $A = (a_{ij}) \in H_{n\times n}$. If λ_i $(i = 1, 2, \ldots, n)$ are n left eigenvalues of A which lie within n distinct generalized spherical neighbourhoods, respectively, then the inequality $\sum_{i=1}^{n} N(\lambda_i) \le \sqrt{n-1}\sum_{i=1}^{n}\left(\sqrt{\sum_{\substack{j=1\\ j\ne i}}^{n} N^2(a_{ij})}\right) + \sum_{i=1}^{n} N(a_{ii})$ holds.*

Proof. Since λ_i $(i = 1, 2, \ldots, n)$ are n left eigenvalues of A which lie within n distinct generalized spherical neighbourhoods, respectively, we can suppose that $\lambda_i \in \Omega_i(A) = \{z : N(z - a_{ii}) \le \sqrt{n-1}R_i\}$, where $R_i = \sqrt{\sum_{\substack{j=1\\ j\ne i}}^{n} N^2(a_{ij})}$ and $\Omega_i(A) \ne \Omega_j(A)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, $i \ne j$.

Then, by Lemma 3.2, $N(\lambda_i - a_{ii}) \le \sqrt{(n-1)\sum_{\substack{j=1\\ j\ne i}}^{n} N^2(a_{ij})}$ $(i = 1, 2, \ldots, n)$. So, we have $N(\lambda_i) = N(\lambda_i - a_{ii} + a_{ii}) \le N(\lambda_i - a_{ii}) + N(a_{ii}) \le \sqrt{n-1}\left(\sqrt{\sum_{\substack{j=1\\ j\ne i}}^{n} N^2(a_{ij})}\right) + N(a_{ii})$, $i = 1, 2, \ldots, n$. So, $\sum_{i=1}^{n} N(\lambda_i) \le \sqrt{n-1}\sum_{i=1}^{n}\left(\sqrt{\sum_{\substack{j=1\\ j\ne i}}^{n} N^2(a_{ij})}\right) + \sum_{i=1}^{n} N(a_{ii})$.

Thus, the proof is complete. □

Theorems 3.1 and 3.2 show the fact that the upper bound of the sum of all the norms of n left eigenvalues of a given quaternion matrix can be bounded by the norms of all entries' of the given quaternion matrix.

In the following, we shall give the third form of the upper bounds of all norm's sum of n left eigenvalues of a given quaternion matrix by the particle and centre gravity theorem.

Theorem 3.3. *Let $A = (a_{ij}) \in H_{n\times n}$. If λ_i $(i = 1, 2, \ldots, n)$ are n left eigenvalues of A which lie within n distinct generalized spherical neighbourhoods, respectively, then*

$$\sum_{i=1}^{n} N(\lambda_i) \le \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\ne i}}^{n} N(a_{ij}) + \sum_{i=1}^{n}\left[N\left(a_{ii} - \frac{\operatorname{tr} A}{n}\right)\right] + N(\operatorname{tr} A).$$

Proof. We assume that λ_i is the ith left eigenvalue of A which lies within ith generalized spherical neighbourhood $\Omega_i(A) = \{z : N(z - a_{ii}) \le R_i\}$, where $R_i = \sum_{\substack{j=1\\ j\ne i}}^{n} N(a_{ij})$, then according to Lemma 3.1, we have $N(\lambda_i - a_{ii}) \le R_i$ (where $R_i = \sum_{\substack{j=1\\ j\ne i}}^{n} N(a_{ij})$ $(i = 1, 2, \ldots, n)$, a_{ii} is the centre of ith generalized spherical neighbourhood). Based on the particle and centre gravity theorem, each $\Omega_i(A)$ $(i = 1, 2, \ldots, n)$ can be treated as a particle or a rigid body. Then the centre of all particles or rigid bodies is $\frac{1}{n}\sum_{i=1}^{n} a_{ii} = \frac{\operatorname{tr} A}{n}$. We have

$$N\left(\lambda_i - \frac{\operatorname{tr} A}{n}\right) = N\left(\lambda_i - a_{ii} + a_{ii} - \frac{\operatorname{tr} A}{n}\right) \le N(\lambda_i - a_{ii}) + N\left(a_{ii} - \frac{\operatorname{tr} A}{n}\right) \le R_i + N\left(a_{ii} - \frac{\operatorname{tr} A}{n}\right)$$

and

$$\begin{aligned} N(\lambda_i) &= N\left(\lambda_i - \frac{\operatorname{tr} A}{n} + \frac{\operatorname{tr} A}{n}\right) \le N\left(\lambda_i - \frac{\operatorname{tr} A}{n}\right) + N\left(\frac{\operatorname{tr} A}{n}\right) \\ &\le R_i + N\left(a_{ii} - \frac{\operatorname{tr} A}{n}\right) + N\left(\frac{\operatorname{tr} A}{n}\right), \quad (i = 1, 2, \ldots, n). \end{aligned}$$

So,

$$\sum_{i=1}^{n} N(\lambda_i) \le \sum_{i=1}^{n}\left[R_i + N\left(a_{ii} - \frac{\operatorname{tr} A}{n}\right) + N\left(\frac{\operatorname{tr} A}{n}\right)\right] = \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\ne i}}^{n} N(a_{ij}) + \sum_{i=1}^{n}\left[N\left(a_{ii} - \frac{\operatorname{tr} A}{n}\right)\right] + N(\operatorname{tr} A).$$

Thus, the proof is complete. □

Next, we give an example to show the correctness of our conclusions.

Example. Let $A = \begin{bmatrix} 1 & k & 0 \\ 0 & i & 0 \\ 0 & 0 & j \end{bmatrix}$. It is clear that 1, i and j are three left eigenvalues of A and we can easily see that $\sum_{i=1}^{3} N(\lambda_i) = 3 \leq \sum_{i=1}^{3}\sum_{j=1}^{3} N(a_{ij}) = 4$. So, the Theorem 3.1 is verified.

Similarly, we have

$$\sum_{i=1}^{3} N(\lambda_i) = 3 \leq \sqrt{3-1}(1+0+0) + 3 = 3 + \sqrt{2} = \sqrt{3-1}\sum_{i=1}^{3}\left(\sqrt{\sum_{\substack{j=1\\ j\neq i}}^{3} N^2(a_{ij})}\right) + \sum_{i=1}^{3} N(a_{ii})$$

and $\sum_{i=1}^{3} N(\lambda_i) = 3 \leq 1 + \sqrt{3} + \sqrt{6} = \sum_{i=1}^{3}\sum_{j\neq i}^{3} N(a_{ij}) + \sum_{i=1}^{3} N(a_{ii} - \frac{\mathrm{tr}A}{3}) + N(\mathrm{tr}A)$, so, Theorems 3.2 and 3.3 also hold.

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